Lecture 4

The one about FIBRATIONS*

Last time. $\mathrm{Flag}(V,\underline{d}) \cong \mathrm{Aut}(V)/P^{abs}_{\underline{d},E} \cong GL_n(k)/P_{\underline{d},E}$

base flag E; bases V; $P_{\underline{d},E}$ block upper tri

$$\cong \begin{cases} U(n)/\prod U(d_i) & k=\mathbb{C} \\ O(n)/\prod O(d_i) & k=\mathbb{R} \end{cases}$$

$\cong G/(G\cap P^{abs}_{\underline{d},E})$ if $G \subset \mathrm{Aut}(V)$ closed, acts transitively on $\mathrm{Flag}(V,\underline{d})$

Cor. $\mathrm{Flag}(V,\underline{d})$ is compact and connected.

Prop. If $k=\mathbb{C}$, $\mathrm{Flag}(V,\underline{d})$ is simply connected.

This will involve some useful ideas.

G Lie grp, $H\subset G$ embedded Lie sub.
Then each fiber of the submersion $G \xrightarrow{\pi} G/H$ is a coset of H, hence an emb. submanifold of G.
These fit together nicely: $\forall x\in G/H$ $\exists$ nbd $\mathcal{U}$ s.t. $\pi^{-1}(\mathcal{U})$ is diffeo to $\mathcal{U}\times H$ in such a way that we get diag

$$\begin{array}{ccc} (E\supset)\ \pi^{-1}(\mathcal{U}) & \xrightarrow[\sim]{\Phi} & \mathcal{U}\times H\ (\mathcal{U}\times F) \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle proj_1} & \\ & \mathcal{U}\ (\subset B) & \end{array}$$

Really a conseq of the SLICE THEOREM

That is, $G\to G/H$ is a locally trivial fiber bdl w/ typical fiber H. I will indicate this by drawing

$$\begin{array}{c} H \to G \\ \downarrow \\ G/H \end{array} \quad \text{means} \quad \begin{array}{c} F \xrightarrow{*} E \\ \downarrow \\ B \end{array}$$

fiber; total sp.; projection (surj); base

* injection, not unique.

** I keep promising Lie theory but remembering more stuff we need first!

Now for a locally trivial fibration, have LES (Locally trivial is enough but also works whenever homotopies lift.)

$$* \left\{ \cdots \to \pi_1(F) \to \pi_1(E) \to \pi_1(B) \to \pi_0(F) \to \pi_0(E) \to \pi_0(B) \to 0 \right.$$

Apply to $\mathcal{U}(n)/\prod \mathcal{U}(d_i)$:

$$\cdots \to \pi_1(\prod \mathcal{U}(d_i)) \longrightarrow \pi_1(\mathcal{U}(n)) \to \pi_1(\mathrm{Flag}) \to \pi_0(\prod \mathcal{U}(d_i))$$

Claim. $\forall\, m \in \mathbb{Z}^+$, $\mathcal{U}(m)$ is connected and $\pi_1(\mathcal{U}(m)) \cong \mathbb{Z}$.
The inclusion of $\mathcal{U}(m') \to \mathcal{U}(m)$ as a block $\begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & \boxed{u} & & \\ & & & 1 & \\ & & & & \ddots \end{pmatrix}$
induces an iso $\pi_1(\mathcal{U}(m')) \cong \pi_1(\mathcal{U}(m))$.

Pf. $\mathcal{U}(n)$ acts on unit sphere $S^{2n-1} \subset \mathbb{C}^n$

This action is transitive. Stabilizer of $\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ is $\begin{pmatrix} 1 & \\ & \boxed{\mathcal{U}(n-1)} \end{pmatrix}$
$\cong \mathcal{U}(n-1)$.

So: $\mathcal{U}(n)$ is a locally trivial bdl over S^{2n-1} w/ fiber $\mathcal{U}(n-1)$.

$\mathcal{U}(1)$ is connected, S^{2n-1} connected $(n \geq 1)$, so total space conn by $*$ at π_0. Induction

Using that S^{2n-1} is simply conn for $n > 1$ we get from π_1 level that $\pi_1(\mathcal{U}(1)) \cong \pi_1(\mathcal{U}(2)) \cong \cdots \cong \pi_1(\mathcal{U}(n))$. $**$

Any desired block on the diag can be made an intermed step in $**$. $\square$

Thus

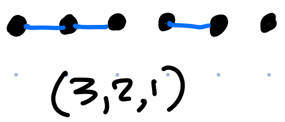
$$\begin{array}{ccccccc} \cdots \to \pi_1(\prod \mathcal{U}(d_i)) & \longrightarrow & \pi_1(\mathcal{U}(n)) & \twoheadrightarrow & \pi_1(\mathrm{Flag}) & \to & \pi_0(\prod \mathcal{U}(d_i)) \\ \prod \mathbb{Z} & \xrightarrow[\text{surj}]{} & \mathbb{Z} & & & & \{0\} \end{array}$$

Hence $\pi_1(\mathrm{Flag}) = 0$.

$\square$

Fibrations between flag vars

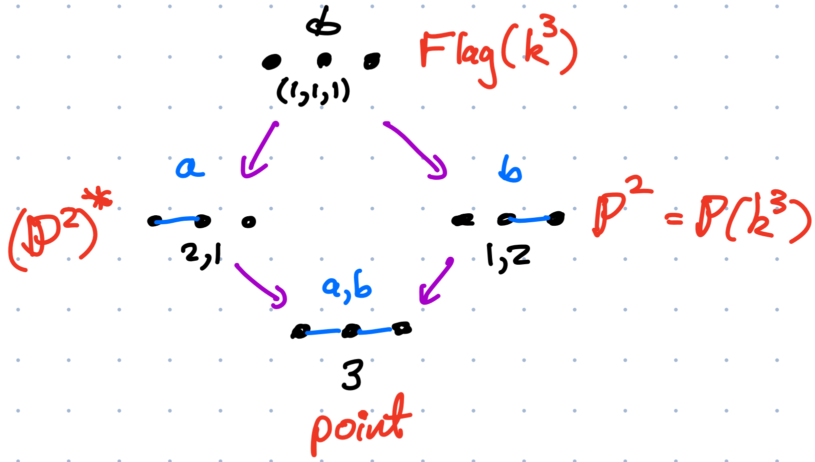
There's a natural partial order on types $\underline{d}$.



type $(d_1 \dots d_m) \to$ diagram of n dots.
first d_1 are conn in a chain
next d_2 as well
...
$\to$ subset of $\{1, \dots, n-1\}$
(which — are present)

Now the subsets of $\{1, \dots, n-1\}$ are ordered by <u>inclusion</u>.

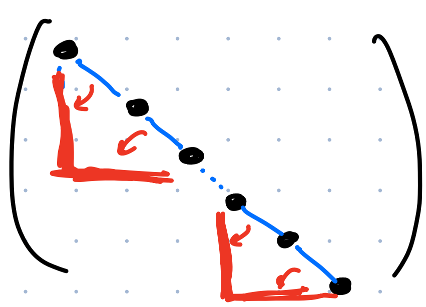
e.g. $n=3$



Lem. Inclusions of sets corresp to $Aut(V)$-equivariant locally trivial fibrations between flag varieties. The fibers are compact.

Pf. Edge set inclusion $A \subset B$ corresponds to the flag varieties being $GL_n(k)/P_A$ and $GL_n(k)/P_B$ with $P_A \subset P_B$.

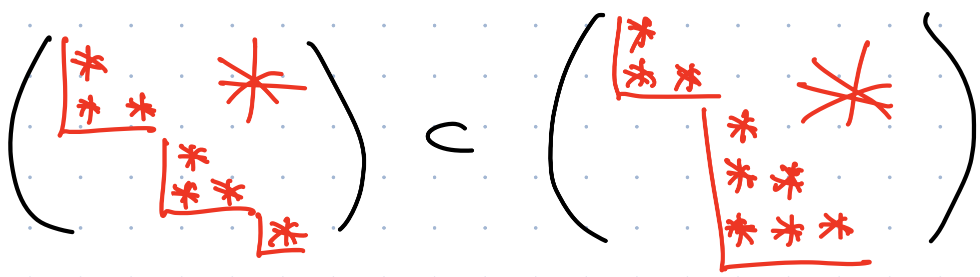
Why?



The blocks are formed by "pushing out" runs of edges along diag to a left + bottom edge.

Thus, an inclusion like

maps to



Now, $G/P_A \to G/P_B$ is well-def + equivt. Also a locally trivial
$gP_A \longmapsto gP_B$

fibration with fiber P_B/P_A (generalizing case $P_A = \{e\}$).

Why compact fiber? Closed subset of G/P_A which is cpt. $\square$

e.g. Every flag var of V has $\mathrm{Flag}(V)$ as a bundle over it.

Closer inspection. The fiber is itself a product of flag varieties.

e.g. $\mathrm{Flag}(a, b, n-(a+b)) \longrightarrow Gr(V, a+b)$

$= \{F_a \subset F_{a+b}\} \longmapsto F_{a+b}$

The fiber over W is $Gr(W, a)$

What about map to $Gr(V,a)$?

Thm. $\mathrm{Flag}(V, \underline{d})$ is a projective variety (subset of $\mathbb{P}^N$ for some N cut out by poly eqn)

Sketch. Realize it as a tower of fibrations where at each step the fiber is a Grassmannian.

$$\mathrm{Flag}(V,\underline{d}) \longrightarrow \mathrm{Flag}(V, \underline{d}^1) \to \cdots \to \underbrace{\mathrm{Flag}(V, d, n-d)}_{Gr(V,d)}$$

At each step you have an ind. hypoth:

$$\mathrm{Flag}(V, \underline{d}^i) \text{ is a proj subvar of } \prod Gr(V, k_i)$$

At the next step, the flag var is cut out by one incidence relation like $F_i \subset F_{i+1}$. Exercise. Incidence var in $Gr \times Gr$ is algebraic. $\square$